NÉRON MODELS OF GREEN-GRIFFITHS-KERR AND LOG NÉRON MODELS

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Abstract. For a variation of Hodge structure over a punctured disk, Green, Griffiths and Kerr introduced a Néron model which is a Hausdorff space that includes values of admissible normal functions. On the other hand, Kato, Nakayama and Usui introduced a Néron model as a logarithmic manifold using log mixed Hodge theory. This work constructs a homeomorphism between these two models.

1. Introduction

Let $J \to \Delta^*$ be a family of intermediate Jacobians arising from a variation of polarized Hodge structure (VHS) of weight -1 with a unipotent monodromy on a punctured disk. By Carlson [C], the intermediate Jacobians are isomorphic to the extension groups of the Hodge structures, in the category of mixed Hodge structures (MHS). Then a section of $J \to \Delta^*$ is known as a variation of MHS (VMHS). A VMHS satisfying the *admissibility condition* [SZ] is called an admissible VMHS (AVMHS) and a section which gives an AVMHS is known as an admissible normal function (ANF) [Sa1].

For the VHS, Green, Griffiths and Kerr [GGK1] introduced the family $J^{\mathrm{GGK}} \to \Delta$ satisfying the following conditions:

- The family restricted to Δ^* is $J \to \Delta^*$;
- The fiber over 0 is a complex Lie group;
- Any ANF is a section of $J^{\mathrm{GGK}} \to \Delta$;
- J^{GGK} is a Hausdorff space.

The total space J^{GGK} is called a *Néron model*. Here, J^{GGK} is simply a topological space. The authors of [GGK1] propose that "One may 'do geometry'" on Néron models.

In contrast, Kato, Nakayama and Usui constructed Néron models via a log mixed Hodge theory. To explain their work, we describe $J \to \Delta^*$ by another formulation. Let $\Delta^* \to \Gamma\backslash D$ be the period map arising from the VHS. The family of intermediate Jacobians can then be obtained as the fiber product:

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Gamma'\backslash D' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta^* \arrow[r] & \Gamma\backslash D \end{tikzcd}$$

where D' and Γ' are used for the MHS corresponding to the intermediate Jacobians.

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Kato, Nakayama and Usui [KNU1] extended the above diagram. First, Kato and Usui [KU] stated that the period map can be extended to

$$\begin{array}{ccc} \Delta & \longrightarrow & \Gamma\backslash D_\Sigma \\ \cup & & \cup \\ \Delta^* & \longrightarrow & \Gamma\backslash D \end{array}$$

where Σ is the fan of nilpotent cones arising from the monodromy of the VHS. Here a boundary point of $\Gamma\backslash D_\Sigma$ is a *nilpotent orbit*, which approximates the period map given by Schmid [Sc]. The main theorem of [KU] states that $\Gamma\backslash D_\Sigma$ is a logarithmic manifold and that it is a moduli space of log (pure) Hodge structures.

Next, an ANF is written as

$$\Delta^* \to \Gamma'\backslash D'.$$

Kato, Nakayama and Usui [KNU2] gives the fan Σ', by which this map can be extended to

$$\begin{array}{ccc} \Delta & \longrightarrow & \Gamma'\backslash D'_{\Sigma'} \\ \cup & & \cup \\ \Delta^* & \longrightarrow & \Gamma'\backslash D' \end{array}.$$

Similarly in the pure case [KU], a boundary point of $\Gamma'\backslash D'_{\Sigma'}$ is a *nilpotent orbit*, which approximates the ANF by the method proposed by Pearlstein [P]. The main theorem of [KNU2] states that $\Gamma'\backslash D'_{\Sigma'}$ is a logarithmic manifold and a moduli space of log mixed Hodge structures.

Finally, they define the *log Néron model* J^{KNU} as the fiber product

$$\begin{array}{ccc} J^{\mathrm{KNU}} & \longrightarrow & \Gamma'\backslash D'_{\Sigma'} \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Gr}^W_{-1}} \\ \Delta & \longrightarrow & \Gamma\backslash D_\Sigma \end{array}$$

in the category of logarithmic manifolds. We remark that J^{KNU} is not only a topological space but also has a geometric structure as a *logarithmic manifold*.

However, [KNU1] does not show the relationship between J^{GGK} and J^{KNU}. In fact, §8.2 of [KNU1] states that the relationship is apparently unknown between J^{KNU} and the Néron model constructed by Green, Griffiths and Kerr. Our main aim is to solve this problem.

Theorem 1.1 (Theorem 5.1)**.** *J^{GGK} is homeomorphic to J^{KNU}.*

We explain the key of the proof. By using the liftings in Equations (4.1) and (4.6), we construct the bijective map between them (in Proposition 4.4 of this paper). In section 5, we show that this map is a homeomorphism. The diagram in Equation (3.5) and the admissibility condition (Equations (2.6) or (2.10)) play important roles in the proof.

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2. Preliminary

In this section, we recall the definitions of the Néron models given in [GGK1] and [KNU2]. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}, \nabla)$ be a variation of polarized Hodge structure of weight -1 over a punctured disk Δ^*, where $\mathcal{H}_{\mathbb{Z}}$ is a local system, $\mathcal{F}$ is a filtration of a locally free sheaf $\mathcal{H} := \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}$ and ∇ is a Gauss-Manin connection. We assume that the monodromy transformation T is unipotent.

2.1. **Families of intermediate Jacobians.** Let (H, F) be the total space of the vector bundle corresponding to the VHS $(\mathcal{H}, \mathcal{F})$. The intermediate Jacobian over $s \in \Delta^*$ is defined as

$$J_s := F_s^0 \backslash H_s / \mathcal{H}_{\mathbb{Z};s}$$

where the subscript s denotes the fiber (or stalk) over s. By Carlson [C], we have the isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H_s) \cong J_s \tag{2.1}$$

where $\mathbb{Z}(0)$ is the Tate's Hodge structure.

We describe the family of intermediate Jacobians $J \to \Delta^*$ using the MHS in (2.1). Fix a reference point $s_0 \in \Delta^*$. For the PHS $H_{s_0} = (H_{\mathbb{Z}}, F_{s_0}, \langle\ ,\ \rangle)$ over s_0, we take a MHS H' which represents an extension class in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H_{s_0})$.ontinuation Let D (resp. D') be the period domain for the type of H_{s_0} (resp. H'), defined in [G] (resp. [U]). The VHS gives the period map $\phi : \Delta^* \to \Gamma\backslash D$ where Γ is the monodromy group. Here we may write

$$\Gamma = \{T^n \in \mathrm{Aut}(H_{\mathbb{Z}}) \mid n \in \mathbb{Z}\}.$$

Then the family of intermediate Jacobians is obtained by the following Cartesian diagram:

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Gamma'\backslash D' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta^* \arrow[r, "\phi"] & \Gamma\backslash D \end{tikzcd}$$

where $\Gamma' := \{T' \in \mathrm{Aut}(H'_{\mathbb{Z}}) \mid T'|_{\mathrm{Aut}(H_{\mathbb{Z}})} \in \Gamma\}$.

We now review some properties of the period domains D and D'. Let $\check{D}$ (resp. $\check{D}'$) be the compact dual of D (resp. D'), defined in [G] (resp. [U]). From [G, §4] (resp. [U, §2]) we have the following properties for the pure case (resp. for some mixed case including the case of D'):

Proposition 2.1. *Let $G_A := \mathrm{Aut}(H_A, \langle\ ,\ \rangle)$ (resp. $G'_A := \mathrm{Aut}(H'_A, \langle\ ,\ \rangle_{\bullet})$) for $A = \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$. Then*

1. *$G_{\mathbb{R}}$ (resp. $G'_{\mathbb{R}}$) acts on D (resp. D') transitively;*
2. *$G_{\mathbb{C}}$ (resp. $G'_{\mathbb{C}}$) acts on $\check{D}$ (resp. $\check{D}'$) transitively;*
3. *Any subgroup of $G_{\mathbb{Z}}$ (resp. $G'_{\mathbb{Z}}$) acts on D (resp. D') properly discontinuously.*

Since H' is an extension of H_{s_0} by $\mathbb{Z}(0)$, we have the exact sequence

$$0 \to H_{\mathbb{Z}} \xrightarrow{i} H'_{\mathbb{Z}} \xrightarrow{j} \mathbb{Z} \to 0$$

of $\mathbb{Z}$-modules. We fix $e \in H'_{\mathbb{Z}}$ such that $j(e) = 1$. Then we may write

$$H'_{\mathbb{Z}} \cong H_{\mathbb{Z}} \oplus \mathbb{Z}e. \tag{2.2}$$

We set

$$\mathfrak{h} := \{X \in \mathrm{End}(H'_{\mathbb{C}}) \mid X|_{\mathrm{End}(H_{\mathbb{C}})} = 0,\ X(e) \in H_{\mathbb{C}}\}.$$

Proposition 2.2 ([U] Theorem 2.16)**.** $\mathrm{Gr}^W_{-1} : \check{D}' \to \check{D}$ *is a fiber bundle with fiber $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{b})$. Here, $\mathfrak{b}$ is the Lie algebra of an isotropy subgroup of $G_{\mathbb{C}}$.*

2.2. **Normal functions and the identity components.** We first define the following sheaves over Δ^*:

$$\mathcal{J} := \mathcal{F}^0\backslash\mathcal{H}/\mathcal{H}_{\mathbb{Z}},$$
$$\mathcal{J}_\nabla := \left\{ \nu \in \mathcal{J} \;\middle|\; \begin{array}{l} \nabla\tilde{\nu} \in \mathcal{F}^{-1}\otimes\Omega^1 \\ \text{for any local lifting } \tilde{\nu} \end{array} \right\}.$$

Since the monodromy is unipotent, we have the Deligne extension $(\mathcal{H}_e, \mathcal{F}_e)$. Then we define the following sheaves over Δ:

$$\mathcal{J}_e := \mathcal{F}_e^0\backslash\mathcal{H}_e/j_*\mathcal{H}_{\mathbb{Z}}, \quad \mathcal{J}_{e,\nabla} := \mathcal{J}_e \cap j_*\mathcal{J}_\nabla$$

where $j : \Delta^* \hookrightarrow \Delta$. A section of $\mathcal{J}_{e,\nabla}$ is called a *normal function* (NF).

Secondly we define a space that includes values of NF according to [GGK1, §II.A]. Let (H_e, F_e) be the total space of the vector bundles corresponding to $(\mathcal{H}_e, \mathcal{F}_e)$. Since these vector bundles are trivial, we have a trivialization

$$F_e^n \cong \Delta \times F_{e;0}^n. \tag{2.3}$$

Since $(F_{e;0}, W(N))$ is a MHS [Sc], we have the Deligne decomposition $H_{e;0} = \bigoplus_{p,q} I^{p,q}$. This decomposition induces

$$F_{e;0}^0\backslash H_{e;0} \cong \bigoplus_{p<0} I^{p,q} =: V. \tag{2.4}$$

By the trivialization (2.3), we may write

$$F_e^0\backslash H_e \cong \Delta \times V.$$

We define the quotient space

$$J^Z := F_e^0\backslash H_e/\sim$$

where the equivalence relation $\sim$ is given by equating two elements $(s,x),(s',x') \in \Delta \times V \cong F_e^0\backslash H_e$ if and only if $s = s'$ and $x - x' \in j_*\mathcal{H}_{\mathbb{Z};s}$. We call it the Zucker space.

The Zucker space J^Z includes values of NF. However, J^Z is *not* generally a Hausdorff space (cf. [GGK1, II.B.8]). Hence, [GGK1] defines the subspace of J^Z so that it is a Hausdorff space including values of NF.

Definition 2.3 ([GGK1] II.A.9)**.** Let

$$W := \{(s,x) \in \Delta \times V \mid x \in \mathrm{Ker}\,(N) \text{ if } s = 0\}. \tag{2.5}$$

The *identity component of the Néron model* is the subset $J^{\mathrm{GGK},0} := W/\sim$ of the Zucker space J^Z. Here the topology on $J^{\mathrm{GGK},0}$ is induced from the *strong topology* of W in $\Delta \times V$ [KU, §3.1.1].

The identity component has the following property:

Proposition 2.4 ([GGK1] II.A.9)**.** *For a NF ν, $\nu(0) \in J_0^{\mathrm{GGK},0}$.*

Remark 2.5. In [GGK1], the definition of the topology on $J^{\mathrm{GGK},0}$ seems to be unclear (a remark after [GGK1, Theorem II.A.9] states “This topology is modeled on the ‘strong topology’ in [KU]”). In this paper, we use the strong topology on $W \subset \Delta \times V$. Saito [Sa2] shows the Hausdorff property in the case of the ordinary topology.

2.3. **Admissible normal functions and Néron models.** In accord with [GGK1, §II.B], we define the sheaf

(2.6)
$$\tilde{\mathcal{J}}_{e,\nabla} := \left\{ \nu \in j_*\mathcal{J}_\nabla \;\middle|\; \begin{array}{l} \tilde{\nu} \text{ has a logarithmic growth as a section of } \check{\mathcal{F}}^0_e, \\ (T-I)\tilde{\nu} \in (T-I)\mathcal{H}_{\mathbb{Q}} \cap \mathcal{H}_{\mathbb{Z}} \text{ for any local lifting } \tilde{\nu}. \end{array} \right\}$$

where we denote the analytic continuation around the origin 0 of $\tilde{\nu}$ by $(T-I)\tilde{\nu}$. A section of $\tilde{\mathcal{J}}_{e,\nabla}$ is called an *admissible normal function* (ANF). By definition, we have the following exact sequence of sheaves:

$$0 \to \mathcal{J}_{e,\nabla} \xrightarrow{i} \tilde{\mathcal{J}}_{e,\nabla} \xrightarrow{j} G_0 \to 0. \tag{2.7}$$

Here G_0 is the skyscraper sheaf supported at 0, whose stalk is

$$G := \frac{(T-I)H_{\mathbb{Q}} \cap H_{\mathbb{Z}}}{(T-I)H_{\mathbb{Z}}}.$$

We define the abelian group

$$J_s^{\mathrm{GGK}} := \frac{J_s^{\mathrm{GGK},0} \times \tilde{\mathcal{J}}_{e,\nabla;s}}{\{(\nu(s), [i(\nu)]_s) \mid \nu \in \mathcal{J}_{e,\nabla}\}}$$

where $[i(\nu)]_s$ is the germ at $s \in \Delta$. Since $\mathcal{J}_{e,\nabla;s}$ is a divisible abelian group (i.e., for every positive integer n and every $\nu \in \mathcal{J}_{e,\nabla;s}$ there exists $\mu \in \mathcal{J}_{e,\nabla;s}$ such that $n\mu = \nu$) and G is a finite group, the exact sequence of the stalks of (2.7) is split [GGK1, II.B.11]. Then we have the isomorphism

$$J_s^{\mathrm{GGK}} \cong \begin{cases} J_s^{\mathrm{GGK},0} & \text{if } s \neq 0, \\ J_s^{\mathrm{GGK},0} \times G & \text{if } s = 0. \end{cases}$$

Definition 2.6 ([GGK1] II.B.9)**.** The *Néron model* of Green-Griffiths-Kerr is the topological space

$$J^{\mathrm{GGK}} := \bigsqcup_{s \in \Delta} J_s^{\mathrm{GGK}}.$$

Here the topology on J^{GGK} is defined by the open sets

$$S(\nu) := \left\{((s,x),[\nu]_s) \in J^{\mathrm{GGK}} \mid (s,x) \in S\right\} \tag{2.8}$$

where S is an open set of $J^{\mathrm{GGK},0}$ and ν is an ANF.

Example 2.7 (Classical case)**.** Let $\bar{f} : \bar{E} \to \Delta$ be a degenerating family of elliptic curves of Kodaira-type I_n. For the restriction $f : E \to \Delta^*$, we have the local system $\mathcal{H}_{\mathbb{Z}} := R^1 f_*\mathbb{Z}$ and the filtration $\mathcal{F}^p = R^1 f_*(\Omega^{\geq p}_{E/\Delta^*})$. Here $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F})$ is a VHS over Δ^* with a unipotent monodromy. In this case,

$$J_0^{\mathrm{GGK},0} \cong \mathbb{G}_m, \quad G \cong \mathbb{Z}\backslash n\mathbb{Z}$$

twisting $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F})$ into the VHS of weight -1.

2.4. **A nonclassical example.** We give an example where the Néron model is not an analytic space. Our two sources, [GGK2, §III.A] and [KNU1, §9], deal with special situations of this kind.

Let Y be a singular $K3$ surface (i.e., $\rho(Y) = 20$) and $\bar{f} : \bar{E} \to \Delta$ be a degenerating family of elliptic curves of Kodaira-type I_n. By the Shioda-Inose correspondence [SI], for a transcendental basis $\{t_1, t_2\}$ of $H^2(Y)$, the intersection form is represented as

$$(t_i \cdot t_j)_{i,j} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$, $a, c > 0$ and $b^2 - 4ac < 0$. We assume that $a = m$, where m is a square free positive integer, and that $b = 0$ and $c = 1$. We take a symplectic basis $\{\alpha, \beta\}$ of $H^1(E_s)$ for $s \neq 0$ such that the monodromy action is

$$\alpha \mapsto \alpha + n\beta, \quad \beta \mapsto \beta.$$

Setting

$$e_1 = t_1 \times \alpha, \quad e_2 = t_2 \times \alpha, \quad e_3 = \frac{t_1}{2m} \times \beta, \quad e_4 = \frac{t_2}{2} \times \beta$$

in $H^3(Y \times E_s, \mathbb{Q})$, then the intersection form is represented as

$$(e_i \cdot e_j)_{i,j} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For the family $g := f \circ \mathrm{pr}_2 : Y \times E \to \Delta^*$, we set the local system $\mathcal{H}_\mathbb{Z} \subset R^3 g_* \mathbb{Q}$ such that $\mathcal{H}_{\mathbb{Z},s} = \sum_i \mathbb{Z} e_i$ and the filtration $\mathcal{F}^p$ induced from $R^3 g_*(\Omega^{\geq p}_{Y \times E/\Delta^*})$. Then $(\mathcal{H}_\mathbb{Z}, \mathcal{F})$ is a VHS and a fiber $(\mathcal{H}_{\mathbb{Z},s}, F_s)$ is a PHS of weight -1 where $h^{1,-2} = h^{0,-1} = h^{-1,0} = h^{-2,1} = 1$, twisting it into the VHS of weight -1. The monodromy transformation is written by

$$T = \begin{pmatrix} I_2 & 0 \\ \begin{matrix} 2mn & 0 \\ 0 & 2n \end{matrix} & I_2 \end{pmatrix}.$$

By [KU, §12.3], the limiting MHS is described by the following Hodge diamond:

$$\begin{tikzcd} \overset{(1,-1)}{\bullet} \arrow[d, "N"] & \overset{(-1,1)}{\bullet} \arrow[d, "N"] \\ \overset{(0,-2)}{\bullet} & \overset{(-2,0)}{\bullet} \end{tikzcd}$$

Then

$$J_0^{\mathrm{GGK},0} \cong I^{-2,0}/j_* \mathcal{H}_{\mathbb{Z};0}, \quad G \cong \mathbb{Z}/2mn\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}.$$

In this case, the dimension of $J_0^{\mathrm{GGK},0}$ is smaller than the dimension of a general fiber and J^Z is not a Hausdorff space (cf. [KNU1, §9]).

2.5. **Moduli spaces of log Hodge structures and log Néron models.** Let $\mathfrak{g}_A$ (resp. $\mathfrak{g}'_A$) be the Lie algebra of G_A (resp. G'_A) for $A = \mathbb{R}, \mathbb{C}$. Writing $\sigma = \mathbb{R}_{\geq 0} N$ in $\mathfrak{g}_\mathbb{R}$ with $N = \log(T)$, we set the fan $\Sigma := \{\{0\}, \sigma\}$ and the set

$$D_\Sigma = \{(\sigma, Z) \mid \sigma \in \Sigma,\ Z = \exp(\sigma_\mathbb{C})F \text{ is a } \sigma\text{-nilpotent orbit}\}. \tag{2.9}$$

By [KU], the period map $\phi : \Delta^* \to \Gamma \backslash D$ extends to the *log* period map $\phi : \Delta \to \Gamma \backslash D_\Sigma$.

Following [KNU1], we define the fan

(2.10)

$$\Sigma' := \left\{ \mathbb{R}_{\geq 0} N' \;\middle|\; \begin{array}{l} N' \in \operatorname{End} H'_\mathbb{Q}, N'|_{\operatorname{End} H_\mathbb{Q}} = N, \\ N'(e) = N(a) \text{ for some } a \in H_\mathbb{Q} \text{ such that } (T - I)a \in H_\mathbb{Z} \end{array} \right\}.$$

Proposition 2.8. *Let $\sigma' \in \Sigma'$. Then there exists a generator $N' \in \operatorname{End} H'_\mathbb{Q}$ of σ' such that* $\exp(N') \in \Gamma'$, *and* $\operatorname{Ad}(\gamma)\sigma' \in \Sigma'$ *for $\gamma \in \Gamma'$. Therefore Γ' is strongly compatible with Σ'.*

Proof. By definition, a generator of σ' is written by

$$\begin{pmatrix} N & Na \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition (2.2) for some $a \in H_{\mathbb{Q}}$. Moreover, we may write

$$\Gamma' = \left\{ \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in H_{\mathbb{Z}},\ n \in \mathbb{Z} \right\}.$$

Since $(T-I)a \in H_{\mathbb{Z}}$, we have

$$\exp(N') = \begin{pmatrix} T & (T-I)a \\ 0 & 1 \end{pmatrix} \in \Gamma'.$$

For $\gamma = \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix} \in \Gamma'$, we have

$$\mathrm{Ad}\,(\gamma)N' = \begin{pmatrix} N & N(T^n a - b) \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

Since $(T-I)(T^n a - b) \in H_{\mathbb{Z}}$, it follows that $\mathrm{Ad}\,(\gamma)N' \in \Sigma'$. □

Similarly in (2.9), $D'_{\Sigma'}$ is defined as the set of nilpotent orbits [KNU2, §2.1.3]. Using the above proposition, we define the action

$$\Gamma' \times D'_{\Sigma'} \to D'_{\Sigma'};\ (\gamma, (\sigma', Z)) \mapsto (\mathrm{Ad}\,(\gamma)\sigma', \gamma Z)$$

and the orbit space $\Gamma' \backslash D'_{\Sigma'}$.

The geometric structure on $\Gamma' \backslash D'_{\Sigma'}$ is defined in [KNU2, §2.2.2]. For a nilpotent cone $\sigma' \in \Sigma'$, we set the monoid

$$\Gamma'(\sigma') := \Gamma' \cap \exp(\sigma')$$

and the toric variety

$$\mathrm{toric}_{\sigma'} := \mathrm{Spec}\,(\mathbb{C}[\Gamma'(\sigma')^{\vee}])_{\mathrm{an}} \cong \mathbb{C}.$$

Moreover, we define the analytic space

$$\check{E}'_{\sigma'} := \mathrm{toric}_{\sigma'} \times \check{D}'$$

and the subspace

$$E'_{\sigma'} = \left\{ (s, F) \in \check{E}'_{\sigma'} \;\middle|\; \begin{array}{l} \exp(l(s)N')F \in D' \text{ if } s \neq 0, \\ \exp(\sigma'_{\mathbb{C}})F \text{ is a nilpotent orbit if } s = 0 \end{array} \right\}$$

where $l(s)$ is a branch of $(2\pi i)^{-1}\log(s)$. The topology on $E'_{\sigma'}$ is the strong topology in $\check{E}'_{\sigma'}$. We then have the map

$$E'_{\sigma'} \overset{p'_1}{\to} \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'} \overset{p'_2}{\to} \Gamma' \backslash D'_{\Sigma'};\quad (s, F) \mapsto \begin{cases} (0, \exp(l(s)N')F) & \text{if } s \neq 0, \\ (\sigma', \exp(\sigma'_{\mathbb{C}})F) & \text{if } s = 0. \end{cases}$$

The geometric structure on $\Gamma' \backslash D'_{\Sigma'}$ is induced from $E'_{\sigma'}$ locally through this map. Moreover Kato, Nakayama and Usui announced the following theorem:

Theorem 2.9 ([KNU2] Main Theorem)**.** *Similarly in the pure case (*[KU, Main Theorem]*), the following holds:*

(1) $E'_{\sigma'}, \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ *and* $\Gamma' \backslash D'_{\Sigma'}$ *are logarithmic manifolds;*
(2) $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ *is a* $\sigma'_{\mathbb{C}}$*-torsor;*
(3) $\Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'} \to \Gamma' \backslash D'_{\Sigma'}$ *is locally an isomorphism;*
(4) $\Gamma' \backslash D'_{\Sigma'}$ *is a moduli space of log mixed Hodge structures.*

Definition 2.10 ([KNU1] §7)**.** The *log Néron model* is the fiber product

$$\begin{tikzcd} J^{\mathrm{KNU}} \arrow[r] \arrow[d] & \Gamma' \backslash D'_{\Sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta \arrow[r, "\phi"] & \Gamma \backslash D_{\Sigma} \end{tikzcd} \tag{2.12}$$

in the category $\mathcal{B}(\log)$ [KU, 3.2.4].

We describe the topology on J^{KNU}. We now have the following diagram:

$$\begin{tikzcd} K_{\sigma'} \arrow[r] \arrow[d] & E'_{\sigma'} \arrow[d, "p'_1"] \\ J_{\sigma'} \arrow[r] \arrow[d] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[d, "p'_2"] \\ J^{\mathrm{KNU}} \arrow[r] & \Gamma'\backslash D'_{\Sigma'} \end{tikzcd} \tag{2.13}$$

where $K_{\sigma'}$ and $J_{\sigma'}$ are the fiber products in $\mathcal{B}(\log)$. Here the topology on $K_{\sigma'}$ is the strong topology in $\Delta \times \check{E}'_{\sigma'}$. The topological structures of $J_{\sigma'}$ (resp. J^{KNU}) are induced from $K_{\sigma'}$ through the morphism $K_{\sigma'} \to J_{\sigma'}$ (resp. $K_{\sigma'} \to J^{\mathrm{KNU}}$).

3. The relationship between $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ and $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$

The results of this section can be easily verified using [KNU2]; however the details will be useful in later sections. In the following section, we regard E_σ (resp. $E'_{\sigma'}$) as a topological space whose topology is the strong topology in $\check{E}_\sigma$ (resp. $\check{E}'_{\sigma'}$).

3.1. $\sigma_{\mathbb{C}}$-action on E_σ and $\sigma'_{\mathbb{C}}$-action on $E'_{\sigma'}$. For $\sigma = \mathbb{R}_{\geq 0}N \in \Sigma$, we define the algebraic torus

$$\mathrm{torus}_\sigma := \mathrm{Spec}\,(\mathbb{C}[\Gamma(\sigma)^{\vee\mathrm{gp}}])_{\mathrm{an}} \cong \mathbb{G}_m$$

and the toric variety

$$\mathrm{torus}_\sigma := \mathrm{Spec}\,(\mathbb{C}[\Gamma(\sigma)^{\vee}])_{\mathrm{an}} \cong \mathbb{C}.$$

We then have the surjective map

$$\sigma_{\mathbb{C}} \to \mathrm{torus}_\sigma; \quad wN \mapsto \exp(2\pi\sqrt{-1}w),$$

which induces the action

$$\sigma_{\mathbb{C}} \times \mathrm{toric}_\sigma \to \mathrm{toric}_\sigma; \quad (wN, s) \mapsto \exp(2\pi\sqrt{-1}w)s.$$

For $\sigma' = \mathbb{R}_{\geq 0}N' \in \Sigma'$, the $\sigma'_{\mathbb{C}}$-action on $\mathrm{toric}_{\sigma'}$ is defined similarly.

By the correspondence $N \leftrightarrow N'$ (resp. $\exp(N) \leftrightarrow \exp(N')$), we have the isomorphism $\sigma_{\mathbb{C}} \cong \sigma'_{\mathbb{C}}$ (resp. $\mathrm{toric}_\sigma \cong \mathrm{toric}_{\sigma'}$) and the following commutative diagram:

$$\begin{tikzcd} \sigma'_{\mathbb{C}} \times \mathrm{toric}_{\sigma'} \arrow[r] \arrow[d, "\wr"] & \mathrm{toric}_{\sigma'} \arrow[d, "\wr"] \\ \sigma_{\mathbb{C}} \times \mathrm{toric}_\sigma \arrow[r] & \mathrm{toric}_\sigma. \end{tikzcd} \tag{3.1}$$

Moreover we define the $\sigma_{\mathbb{C}}$-action

$$\sigma_{\mathbb{C}} \times E_\sigma \to E_\sigma; \quad (wN, (s, F)) \mapsto (\exp(2\pi\sqrt{-1}w)s, \exp(-wN)F),$$

and the $\sigma'_{\mathbb{C}}$-action on $E'_{\sigma'}$ is defined similarly. Setting

$$\mathrm{Gr}^W_{-1} : \check{E}'_{\sigma'} \to \check{E}_\sigma; \quad (s, F) \mapsto (s, \mathrm{Gr}^W_{-1}(F)),$$

the diagram (3.1) induces the following commutative diagram:

$$\begin{tikzcd} \sigma'_{\mathbb{C}} \times E'_{\sigma'} \arrow[r] \arrow[d] & E'_{\sigma'} \arrow[d] & \subset & \check{E}'_{\sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \sigma_{\mathbb{C}} \times E_\sigma \arrow[r] & E_\sigma & \subset & \check{E}_\sigma. \end{tikzcd} \tag{3.2}$$

3.2. **The torsor property of $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$.**

Lemma 3.1. *The action of $\sigma'_{\mathbb{C}}$ on $E'_{\sigma'}$ is proper and free.*

Proof. Since the lower horizontal action in (3.2) is free [KU, (7.2.9)], the upper horizontal action in (3.2) is also free.

The $\sigma'_{\mathbb{C}}$-action is proper if and only if the following condition is satisfied:

- For $x', y' \in E'_{\sigma'}$, sequences $\{x'_\lambda\}$ in $E'_{\sigma'}$ and $\{h'_\lambda\}$ in $\sigma'_{\mathbb{C}}$ such that $x'_\lambda \to x'$ and $h'_\lambda x'_\lambda \to y'$, there exists $h' \in \sigma'_{\mathbb{C}}$ such that $h'_\lambda \to h'$.

We will now show that the above condition holds. Taking $x', y', \{x'_\lambda\}, \{h'_\lambda\}$ as above, we let

$$x := \mathrm{Gr}^W_{-1}(x'), \quad y = \mathrm{Gr}^W_{-1}(y'), \quad h_\lambda := h'_\lambda|_{\mathrm{End}\, H_{\mathbb{Q}}}.$$

Since the $\sigma_{\mathbb{C}}$-action is proper [KU, (7.2.2)], there exists $h \in \sigma_{\mathbb{C}}$ such that $h_\lambda \to h$. By the isomorphism $\sigma \cong \sigma'$, there exists $h' \in \sigma'_{\mathbb{C}}$ such that $h = h'|_{\mathrm{End}\, H_{\mathbb{Q}}}$ and $h'_\lambda \to h'$. □

Lemma 3.2 ([KU] Lemma 7.3.3)**.** *Let H be a topological group and X be a topological space, and assume that we have an action $H \times X \to X$ which is proper and free. Furthermore assume that the following condition is satisfied:*

- *For $x \in X$, there exists a topological space S, a morphism $\iota : S \to X$ and an open neighborhood U of 1 in H such that $U \times S \to X$; $(h, s) \mapsto h\iota(s)$ induces an isomorphism onto an open set of X.*

Then $X \to H \backslash X$ is an H-torsor.

Proposition 3.3 ([KNU2] Theorem A.(2))**.** *The action of $\sigma'_{\mathbb{C}}$ on $E'_{\sigma'}$ satisfies the condition of Lemma 3.2. Then $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ is a $\sigma'_{\mathbb{C}}$-torsor.*

Proof. Since $\sigma'(s)_{\mathbb{C}} \hookrightarrow T_{\check{D}'}(F)$ for $(s, F) \in E'_{\sigma'}$ (in this case $\sigma'(s) = \sigma'$ if $s = 0$, and $\sigma'(s) = 0$ otherwise), the proof is the same as for the pure case ([KU, (7.3.5)]). □

Since $p_1 : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ (resp. $p'_1 : E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$) is a $\sigma_{\mathbb{C}}$-torsor ($\sigma'_{\mathbb{C}}$-torsor), the diagram (3.2) induces the following property:

Corollary 3.4. *The commutative diagram*

$$\begin{tikzcd} E'_{\sigma'} \arrow[r, "p'_1"] \arrow[d, "\mathrm{Gr}^W_{-1}"'] & \Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'} \arrow[d] \\ E_\sigma \arrow[r, "p_1"] & \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma \end{tikzcd} \tag{3.3}$$

is Cartesian.

3.3. **Limiting Hodge filtrations and liftings of the period map.**

Let $\tilde{\phi}$ be a local lifting of the period map ϕ. Then we have the holomorphic map

$$\hat{\phi} : \Delta^* \to \check{D}; \quad s \mapsto \exp(-l(s)N)\tilde{\phi}(s). \tag{3.4}$$

We call this an *untwisted period map.* By [Sc], this map is extended over Δ. We denote $\hat{\phi}(0)$ by $F_{\tilde{\phi}}$. Remark that $F_{\tilde{\phi}}$ depends upon the choice of local lifting $\tilde{\phi}$. The untwisted map $\hat{\phi}$ gives a lifting

$$\Delta \to E_\sigma; \quad s \mapsto (s, \hat{\phi}(s))$$

of ϕ. This gives the following diagram:

$$
\begin{tikzcd}
\check{E}'_{\sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"'] & \supset & E'_{\sigma'} \arrow[r, "p'_1"] \arrow[d] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[r, "p'_2"] \arrow[d] & \Gamma'\backslash D'_{\Sigma'} \arrow[d] \\
\check{E}_{\sigma} & \supset & E_{\sigma} \arrow[r, "p_1"] & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma} & \Gamma\backslash D_{\Sigma} \arrow[l, phantom, "="] \\
& & \Delta \arrow[u, "(id{,}\hat{\phi})"] \arrow[ur, "\phi"'] & &
\end{tikzcd} \tag{3.5}
$$

for $\sigma' \in \Sigma'$ such that $\sigma' \neq \{0\}$.

For $(s, F) \in \check{E}'_{\sigma'}$ such that $\mathrm{Gr}^W_{-1}(F) = F_{\hat{\phi}(s)}$, we have the exact sequence

$$0 \to F^p_{\hat{\phi}(s)} \to F^p \to \mathbb{C} \to 0$$

if $p \leq 0$, and $F^p_{\hat{\phi}(s)} \cong F^p$ otherwise. Then

$$F^p = \begin{cases} \mathbb{C}(z,1) + F^p_{\hat{\phi}(s)} & \text{if } p \leq 0, \\ F^p_{\hat{\phi}(s)} & \text{if } p > 0 \end{cases} \tag{3.6}$$

where $(z, 1) \in H'_{\mathbb{C}}$ is represented with respect to the decomposition (2.2). By the admissibility condition (2.10), a generator of $\sigma' \in \Sigma'$ can be written as

$$N' = \begin{pmatrix} N & Na \\ 0 & 0 \end{pmatrix}$$

for some $a \in H_{\mathbb{Q}}$.

Proposition 3.5. *For $(s, F) \in \check{E}'_{\sigma'}$ such that $\mathrm{Gr}^W_{-1}(F) = F_{\hat{\phi}(s)}$,*

$$(s, F) \in E'_{\sigma'} \Longleftrightarrow \begin{cases} z \in H_{\mathbb{C}} & \textit{if } s \neq 0, \\ z + a \in F^0_{\tilde{\phi}} + \mathrm{Ker}\,(N) & \textit{if } s = 0 \end{cases}$$

where $z \in H_{\mathbb{C}}$ is in (3.6).

Proof. If $s \neq 0$, then

$$\mathrm{Gr}^W_{-1}\,(\exp\,(l(s)N')F) = \exp\,(l(s)N)\hat{\phi}(s) = \tilde{\phi}(s) \in D$$

for any $z \in H_{\mathbb{C}}$. Then $(s, F) \in E'_{\sigma'}$ for any $z \in H_{\mathbb{C}}$.

If $s = 0$, then

$$N(z + a) \in F^{-1}_{\tilde{\phi}}$$

by the transversality condition for nilpotent orbits. Since $(F_{\tilde{\phi}}, W(N))$ is a MHS and N is a $(-1,-1)$-morphism, $N(z + a) \in F^{-1}_{\tilde{\phi}}$ if $z + a \in F^0_{\tilde{\phi}} + \mathrm{Ker}\,(N)$. □

4. A bijection

In this section, we define a bijective map between J^{KNU} and J^{GGK} as sets.

4.1. **A map from J^{GGK} to J^{KNU}.** Let ν be an ANF. The ANF ν defines an AVMHS

$$\nu : \Delta^* \to \Gamma'\backslash D'.$$

Taking a local lifting $\tilde{\nu}$ of ν, we have a local lifting $\tilde{\phi} = \mathrm{Gr}^W_{-1}(\tilde{\nu})$ of ϕ. Let N' be the logarithm of monodromy of $\tilde{\nu}$. Similarly in (3.4), we define

$$\hat{\nu} : \Delta^* \to \check{D}'; \quad s \mapsto \exp\,(-l(s)N')\tilde{\nu}(s).$$

By the admissibility condition (2.6), $\hat{\nu}$ extend over Δ and $\sigma' = \mathbb{R}_{\geq 0}N$ is in Σ'. We denote $\hat{\nu}(0)$ by $F_{\tilde{\nu}}$. By [P], $(\sigma', F_{\tilde{\nu}})$ is a nilpotent orbit. We have the commutative diagram

$$\begin{tikzcd} & \check{D}' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta \arrow[ur, "\hat{\nu}"] \arrow[r, "\hat{\phi}"'] & \check{D}, \end{tikzcd} \tag{4.1}$$

that is $\hat{\nu}$ is a lifting of $\hat{\phi}$.

We fix $F_{\tilde{\nu}}$ as a reference point of $\check{D}'$. By Proposition 2.2, the vertical morphism of the above diagram is the fiber bundle with fiber $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{b})$. Recall that

$$\begin{aligned} &\mathfrak{h} = \{X \in \mathrm{End}\,(H'_{\mathbb{C}}) \mid X|_{\mathrm{End}\,(H_{\mathbb{C}})} = 0,\ X(e) \in H_{\mathbb{C}}\}, \\ &\mathfrak{h} \cap \mathfrak{b} = \{X \in \mathfrak{h} \mid X(e) \in F^0_{\tilde{\phi}}\}, \\ &V = \bigoplus_{p<0} I^{p,q} \quad \left(\text{i.e., } F^0_{\tilde{\phi}} \oplus V = H_{\mathbb{C}}\right), \end{aligned}$$

and then

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{b}) \cong V; \quad X_v \leftrightarrow v \tag{4.2}$$

where $X_v \in \mathfrak{h}$ such that $X_v(e) = v$. Taking a boundary point $((0, \dot{v}), [\nu]_0) \in J^{\mathrm{GGK}}$ where

$$\dot{v} = v \mod H_{\mathbb{Z}} \cap \mathrm{Ker}\, N \tag{4.3}$$

for some $v \in V \cap \mathrm{Ker}\,(N)$, we define

$$\alpha((0, \dot{v}), [\nu]_0) := (0, (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v}) F_{\tilde{\nu}})).$$

By Proposition 3.5, $\alpha((0, \dot{v}), [\nu]_0)$ is in J^{KNU}.

Lemma 4.1. *$\alpha((0, \dot{v}), [\nu]_0)$ is well-defined.*

Proof. We show that $\alpha((0, \dot{v}), [\nu]_0)$ does not depend on the choice of v of (4.3), a lifting $\tilde{\nu}$ and a representative $((0, \dot{v}), [\nu]_0)$.

First, we take $x \in H_{\mathbb{Z}} \cap \mathrm{Ker}\,(N)$. By (2.11), this gives $\mathrm{Ad}\,(\gamma_x)N' = N'$ for

$$\gamma_x = \begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix} \in \Gamma'.$$

Then

$$\begin{aligned} (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v+x}) F_{\tilde{\nu}}) &= (\sigma', \exp(\sigma'_{\mathbb{C}}) \gamma_x \exp(X_{-v}) F_{\tilde{\nu}}) \\ &= \gamma_x (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v}) F_{\tilde{\nu}}). \end{aligned}$$

Next, we take another lifting $\gamma\tilde{\nu}$ for $\gamma \in \Gamma'$. The monodromy cone that arises from $\gamma\tilde{\nu}$ is $\mathrm{Ad}\,(\gamma)\sigma'$ and $F_{\gamma\tilde{\nu}} = \gamma F_{\tilde{\nu}}$. Since $v \in \mathrm{Ker}\, N$, we have

$$\exp(X_{-v})\gamma = \gamma \exp(X_{-v}).$$

Then

$$(\mathrm{Ad}\,(\gamma)\sigma'_{\mathbb{C}}, \exp(\mathrm{Ad}\,(\gamma)\sigma'_{\mathbb{C}}) \exp(X_{-v}) F_{\gamma\tilde{\nu}}) = \gamma(\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v}) F_{\tilde{\nu}}).$$

Finally, we take $((0, \dot{v}_1), [\nu_1]_0) \sim ((0, \dot{v}_2), [\nu_2]_0)$ and let

$$F^p_{\tilde{\nu}_i} = \begin{cases} \mathbb{C}(z_i, 1) + F^p_{\tilde{\phi}} & \text{if } p \leq 0, \\ F^p_{\tilde{\phi}} & \text{if } p > 0, \end{cases} \quad \text{for } i = 1, 2,$$

where $\tilde{\nu}_i$ are local liftings. Let $\mu = \nu_1 - \nu_2$. Then there exists a local lifting $\tilde{\mu}$ such that

$$F^p_{\tilde{\mu}} = \begin{cases} \mathbb{C}(z_1 - z_2, 1) + F^p_{\tilde{\phi}} & \text{if } p \leq 0, \\ F^p_{\tilde{\phi}} & \text{if } p > 0. \end{cases}$$

Since $((0, \dot{v}_1), [\nu_1]_0) \sim ((0, \dot{v}_2), [\nu_2]_0)$, μ is a NF such that $\mu(0) = \dot{v}_1 - \dot{v}_2 \in J_0^{\mathrm{GGK},0}$. Then there exists $v_1, v_2 \in \mathrm{Ker}\,(N) \cap V$ such that

$$\dot{v}_i = v_i \mod H_{\mathbb{Z}} \cap \mathrm{Ker}\, N.$$

and $z_1 - z_2 = v_1 - v_2$.

On the other hand, the logarithm of the monodromy of $\tilde{\nu}_i$ is described by

$$\begin{pmatrix} N & Na_i \\ 0 & 0 \end{pmatrix}$$

for some $a_i \in H_{\mathbb{Q}}$. Then the logarithm of the monodromy of $\tilde{\mu}$ is

$$\begin{pmatrix} N & N(a_1 - a_2) \\ 0 & 0 \end{pmatrix}.$$

Since μ is a NF, $(T - I)(a_1 - a_2) \in (T - I)H_{\mathbb{Z}}$ by the exact sequence (2.7). Then $a_1 - a_2 \in H_{\mathbb{Z}}$. Setting

$$\gamma_{a_1 - a_2} = \begin{pmatrix} I & a_1 - a_2 \\ 0 & 1 \end{pmatrix},$$

we have

$$\mathrm{Ad}\,(\gamma_{a_1-a_2}) \begin{pmatrix} N & Na_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} N & Na_1 \\ 0 & 0 \end{pmatrix}$$

by (2.11). Since $\alpha((0, \dot{v_2}), [\nu_2]_0)$ does not depend on the choice of lifting, we may take $\gamma_{a_1-a_2}\tilde{\nu}_2$ as a lifting of ν_2. The monodromy cone that arises from $\gamma_{a_1-a_2}\tilde{\nu}_2$ is σ'. Then

$$\begin{aligned} (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v_2}) F_{\tilde{\nu}_2}) &= (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v_2}) \exp(X_{v_2 - v_1}) F_{\tilde{\nu}_1}) \\ &= (\sigma', \exp(\sigma'_{\mathbb{C}}) \exp(X_{-v_1}) F_{\tilde{\nu}_1}). \end{aligned}$$

□

Therefore, α defines a map

$$\alpha : J^{\mathrm{GGK}} \to J^{\mathrm{KNU}}$$

where the restriction $\alpha|_J$ is canonical.

4.2. **A map from J^{KNU} to J^{GGK}.** Let $\tilde{\phi}$ be a lifting of ϕ. By Corollary 3.4, for $(0, (\sigma', Z)) \in J_{\sigma'}$, we have $(0, F) \in E'_{\sigma'}$ such that

$$\mathrm{Gr}^W_{-1}(0, F) = (0, F_{\tilde{\phi}}), \quad p'_1(0, F) = (\sigma', Z). \tag{4.4}$$

We denote this filtration by $F_{(\sigma', Z), \tilde{\phi}}$.

Lemma 4.2. *For $\gamma \in \Gamma'$ such that $\gamma|_{\mathrm{Aut}\, H_{\mathbb{Z}}} = T^n$, $\gamma \exp((m - n)N') F_{(\sigma', Z), \tilde{\phi}} = F_{\gamma(\sigma', Z), T^m \tilde{\phi}}$.*

Proof. By Proposition 3.5, there exists $x \in \mathrm{Ker}\,(N)$ such that

$$F^p_{(\sigma', Z), \tilde{\phi}} = \begin{cases} \mathbb{C}(x - a, 1) + F^p_{\tilde{\phi}} & \text{if } p \leq 0, \\ F^p_{\tilde{\phi}} & \text{if } p > 0. \end{cases}$$

Writing $\gamma = \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix}$ for some $b \in H_{\mathbb{Z}}$, we have

$$\gamma \exp((m-n)N')F^p_{(\sigma',Z),\tilde{\phi}} = \begin{cases} \mathbb{C}(T^m x - T^n a + b, 1) + F^p_{T^m\tilde{\phi}} & \text{if } p \leq 0, \\ F^p_{T^m\tilde{\phi}} & \text{if } p > 0. \end{cases}$$

Since $x \in \operatorname{Ker}(N)$,

$$T^m x - T^n a + b = x - (T^n a - b). \tag{4.5}$$

By (2.11) and Proposition 3.5, $(0, \gamma \exp((m-n)N')F_{(\sigma',Z),\tilde{\phi}}) \in E'_{\operatorname{Ad}(\gamma)\sigma'}$, which satisfies

$$\begin{aligned} \operatorname{Gr}^W_{-1}(0, \gamma \exp((m-n)N')F_{(\sigma',Z),\tilde{\phi}}) &= (0, F_{T^m\tilde{\phi}}), \\ p'_1(0, \gamma \exp((m-n)N')F_{(\sigma',Z),\tilde{\phi}}) &= \gamma(\sigma', Z). \end{aligned}$$

□

Let $\hat{\phi} : \Delta \to \check{D}$ be the untwisted period map. Since $\check{D}' \to \check{D}$ is a fiber bundle, there exists a lifting of $\hat{\phi}$:

(4.6)
$$\begin{tikzcd} & \check{D}' \arrow[d, "\operatorname{Gr}^W_{-1}"] \\ \Delta \arrow[ur, "\hat{\nu}_{(\sigma',Z),\tilde{\phi}}"] \arrow[r, "\hat{\phi}"] & \check{D} \end{tikzcd}$$

such that $\hat{\nu}_{(\sigma',Z),\tilde{\phi}}(0) = F_{(\sigma',Z),\tilde{\phi}}$, shrinking Δ if necessary. We then have a holomorphic map

$$\Delta^* \to \Gamma' \backslash D'; \quad s \mapsto p'_2 \circ p'_1(s, \hat{\nu}_{(\sigma',Z),\tilde{\phi}}(s)),$$

which defines an AVMHS, i.e., an ANF. Denoting this ANF by $\nu_{(\sigma',Z),\tilde{\phi}}$, we define

$$\beta(0, (\sigma', Z)) := ((0,0), [\nu_{(\sigma',Z),\tilde{\phi}}]_0) \in J^{\mathrm{GGK}}.$$

Lemma 4.3. *$\beta(0, (\sigma', Z))$ is well-defined.*

Proof. We show that $\beta(0, (\sigma', Z))$ does not depend on the choice of $\hat{\nu}_{(\sigma',Z),\tilde{\phi}}$, (σ', Z) and $\tilde{\phi}$.

If we take liftings $\hat{\nu}_{(\sigma',Z),\tilde{\phi}}$ and $\hat{\nu}'_{(\sigma',Z),\tilde{\phi}}$ such that

$$\hat{\nu}_{(\sigma',Z),\tilde{\phi}}(0) = \hat{\nu}'_{(\sigma',Z),\tilde{\phi}}(0) = F_{(\sigma',Z),\tilde{\phi}},$$

then $\mu := \nu_{(\sigma',Z),\tilde{\phi}} - \nu'_{(\sigma',Z),\tilde{\phi}}$ is a NF and $\mu(0) = 0 \in J_0^{\mathrm{GGK},0}$. Then

$$((0,0), [\nu_{(\sigma',Z),\tilde{\phi}}]_0) \sim ((0,0), [\nu'_{(\sigma',Z),\tilde{\phi}}]_0).$$

Moreover, by Lemma 4.2,

$$\gamma \exp((m-n)N')\hat{\nu}_{(\sigma',Z),\tilde{\phi}}(0) = F_{\gamma(\sigma',Z),T^m\tilde{\phi}}.$$

If we take $\hat{\nu}_{\gamma(\sigma',Z),T^m\tilde{\phi}} = \gamma \exp((m-n)N')\hat{\nu}_{(\sigma',Z),\tilde{\phi}}$ as a lifting of $T^m\hat{\phi}$, then $\nu_{(\sigma',Z),\tilde{\phi}} = \nu_{\gamma(\sigma',Z),T^m\tilde{\phi}}$. □

Then β defines a map

$$\beta : J^{\mathrm{KNU}} \to J^{\mathrm{GGK}}$$

where the restriction $\beta|_J$ is canonical.

Proposition 4.4. *$\alpha = \beta^{-1}$ and $\beta = \alpha^{-1}$, i.e., J^{GGK} is bijective to J^{KNU}.*

Proof. For $((0,\dot{v}),[\nu]_0)\in J^{\mathrm{GGK}}$, we set $(0,(\sigma',Z)):=\alpha((0,\dot{v}),[\nu]_0)$. By making suitable choice of $\tilde{\nu}$, $\tilde{\phi}$ and v, we have $F_{(\sigma',Z),\tilde{\phi}}=\exp(X_{-v})F_{\tilde{\nu}}$. Therefore $\mu(0)=\dot{v}$ for $\mu=\nu-\nu_{(\sigma',Z),\tilde{\phi}}$, which induces

$$((0,\dot{v}),[\nu]_0)\sim((0,0),[\nu_{(\sigma',Z),\tilde{\phi}}]_0)=\beta(0,(\sigma',Z)).$$

On the other hand, for $(0,(\sigma',Z))\in J^{\mathrm{KNU}}$, we set $((0,0),[\nu]_0):=\beta(0,(\sigma',Z))$. By making suitable choice of $\tilde{\nu}$, (σ',Z) and $\tilde{\phi}$, we have $F_{\tilde{\nu}}=F_{(\sigma',Z),\tilde{\phi}}$. Therefore,

$$(0,(\sigma',Z))=(0,(\sigma',\exp(\sigma'_{\mathbb{C}})F_{\tilde{\nu}}))=\alpha((0,0),[\nu]_0).$$

□

5. A homeomorphism

In this section, we show the following main theorem

Theorem 5.1. *J^{GGK} is homeomorphic to J^{KNU}.*

To show continuity, we describe an open neighborhood in J^{KNU}. We recall that the topology on J^{KNU} is induced from K_σ through the following diagram:

$$\begin{tikzcd}
K_{\sigma'} \arrow[r] \arrow[d] & E'_{\sigma'} \arrow[d, "p'_1"] \\
J_{\sigma'} \arrow[r] \arrow[d] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[d, "p'_2"] \\
J^{\mathrm{KNU}} \arrow[r] \arrow[d] & \Gamma'\backslash D'_{\Sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\
\Delta \arrow[r, "\phi"] & \Gamma\backslash D_\Sigma.
\end{tikzcd}$$

We describe an open neighborhood in J^{KNU} using the following steps:
Step 1. Describe an open neighborhood in $E'_{\sigma'}$;
Step 2. Describe an open neighborhood in $\Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$; and
Step 3. Describe an open neighborhood in J^{KNU}.
Open neighborhoods in J^{GGK} are described in (2.8). Comparing these, we show that the bijection constructed in the last section is continuous.

5.1. **Proof of the main theorem.**

Setting: We take a boundary point $(0,(\sigma',Z))\in J^{\mathrm{KNU}}$. Setting a lifting $\tilde{\phi}$ of the period map ϕ, we have the untwisted period map $\hat{\phi}:\Delta\to\check{D}$. Since $\sigma_{\mathbb{C}}\hookrightarrow T_{\check{D}}(F_{\tilde{\phi}})$, we may take a $\mathbb{C}$-subspace B of $\mathfrak{g}_{\mathbb{C}}$ such that $B\oplus\sigma_{\mathbb{C}}\cong T_{\check{D}}(F_{\tilde{\phi}})$. An open neighborhood at $F_{\tilde{\phi}}$ in $\check{D}$ is described by

$$\{\exp(a_1)\exp(a_2)F_{\tilde{\phi}}\mid a_1\in U_1,\ a_2\in U_2\}\cong U_1\times U_2$$

where U_1 (resp. U_2) is a sufficiently small open neighborhood of 0 in $\sigma_{\mathbb{C}}$ (resp. B). We assume that the image of $\hat{\phi}$ is included in this open neighborhood shrinking Δ if necessary. We put $\hat{\phi}(s)=(\hat{\phi}_1(s),\hat{\phi}_2(s))$, where $\hat{\phi}_1:\Delta\to\sigma_{\mathbb{C}}\cong\mathbb{C}$ is a holomorphic function such that $\hat{\phi}_1(0)=0$. By using the coordinate $t=\exp(2\pi\sqrt{-1}\hat{\phi}_1(s))\cdot s$ on Δ the untwisted period map is

$$\begin{aligned}\hat{\phi}(t)=\exp(-l(t)N)\tilde{\phi}(t)&=\exp\left(-\hat{\phi}_1(s)N-l(s)N\right)\tilde{\phi}(t)\\&=\exp(-\hat{\phi}_1(s)N)\hat{\phi}(s).\end{aligned}$$

Then $\hat{\phi}_1(t) = 0$ for $\hat{\phi}(t) = (\hat{\phi}_1(t), \hat{\phi}_2(t)) \in U_1 \times U_2$. It is significant that $F_{\tilde{\phi}}$, $F_{\tilde{\nu}}$ and $F_{(\sigma', Z), \tilde{\phi}}$ do not depend on this coordinate change (i.e., the bijection α is independent).

Step 1 and Step 2: In the pure case, neighborhoods in E_σ and in $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ are described in [KU, (7.3.5)]. We describe neighborhoods in $E'_{\sigma'}$ and in $\Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ in a similar way. Now we have the point $(0, F_{(\sigma', Z), \tilde{\phi}}) \in E'_{\sigma'}$ as described in (4.4). Since $\mathrm{Gr}^W_{-1} : \check{D}' \to \check{D}$ is a fiber bundle whose fiber is V, we have a local trivialization

$$(\mathrm{Gr}^W_{-1})^{-1}(U_1 \times U_2) \cong U_1 \times U_2 \times V. \tag{5.1}$$

Since $F_{(\sigma', Z), \tilde{\phi}} \in (\mathrm{Gr}^W_{-1})^{-1}(F_{\tilde{\phi}})$, we can assume that $(0, 0, 0)$ corresponds to $F_{(\sigma', Z), \tilde{\phi}}$. Using this local trivialization, an open neighborhood at $(0, F_{(\sigma', Z), \tilde{\phi}})$ in $\check{E}'_{\sigma'}$ can be described by

$$\{(a_0, (a_1, a_2, v)) \mid a_0 \in U_0,\ a_1 \in U_1,\ a_2 \in U_2,\ v \in U_3\}$$

where U_0 (resp. U_3) is a sufficiently small open neighborhood of 0 in $\mathrm{toric}_{\sigma'}$ (resp. V). Let

$$A' = \{(a_0, (0, a_2, v)) \mid a_0 \in U_0,\ a_2 \in U_2,\ v \in U_3\}, \quad S' = A' \cap E'_{\sigma'}.$$

Using the diagram (3.2), the $\sigma'_{\mathbb{C}}$-action defines an open inclusion map

$$U_1 \times S' \hookrightarrow E'_{\sigma'}.$$

This inclusion map induces the open inclusion map

$$\sigma'_{\mathbb{C}} \times S' \hookrightarrow E'_{\sigma'},$$

shrinking S' if necessary. Then $p'_1(S')$ is an open set of $\Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ and $p'_1(S') \cong S'$. Moreover, $p'_2 \circ p'_1(S')$ is an open neighborhood of (σ', Z) in $\Gamma' \backslash D'_{\Sigma'}$.

Step 3: Since $p'_1(S')$ (resp. $p'_2 \circ p'_1(S')$) is an open neighborhood in $\Gamma'(\sigma')^{\mathrm{gp}} \backslash D'_{\sigma'}$ (resp. $\Gamma' \backslash D'_{\Sigma'}$), $p'_1((\Delta \times S') \cap K_{\sigma'})$ (resp. $p'_2 \circ p'_1((\Delta \times S') \cap K_{\sigma'})$) is an open neighborhood of $J_{\sigma'}$ (resp. J^{KNU}). Moreover, since $p'_1(S') \cong S'$,

$$p'_1((\Delta \times S') \cap K_{\sigma'}) \cong (\Delta \times S') \cap K_{\sigma'}.$$

We describe $(\Delta \times S') \cap K_{\sigma'}$ explicitly. By the diagram (3.5), we have the following commutative diagram:

$$\begin{tikzcd}
K_{\sigma'} \arrow[rr] \arrow[dd] & & E'_{\sigma'} \arrow[dl, "\mathrm{Gr}^W_{-1}"'] \arrow[d, "p'_2 \circ p'_1"] \\
& E_\sigma \arrow[dr, "p_1"] & \Gamma' \backslash D'_{\Sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\
\Delta \arrow[rr, "\phi"] & & \Gamma \backslash D_\Sigma.
\end{tikzcd} \tag{5.2}$$

Then, for $(t, \xi) \in \Delta \times E'_{\sigma'}$, $(t, \xi) \in K_{\sigma'}$ if, and only if,

$$\phi(t) = \mathrm{Gr}^W_{-1} \circ p'_2 \circ p'_1(\xi) = p_1 \circ \mathrm{Gr}^W_{-1}(\xi).$$

Lemma 5.2. $\big((p_1)^{-1}(\phi(t))\big) \cap \mathrm{Gr}^W_{-1}(S') = (t, \hat{\phi}(t))$.

Proof. Since $p_1((t, \hat{\phi}(t))) = \phi(t)$ and p_1 is a $\sigma_{\mathbb{C}}$-torsor, the fiber is

$$(p_1)^{-1}(\phi(t)) = \sigma_{\mathbb{C}} \cdot (t, \hat{\phi}(t)) = \{(\exp(2\pi\sqrt{-1}x)t, \exp(-xN)\hat{\phi}(t)) \mid x \in \mathbb{C}\}.$$

The intersection with $U_0 \times U_1 \times U_2$ is

$$\begin{aligned}&(U_0 \times U_1 \times U_2) \cap (p_1)^{-1}(\phi(t))\\&= \{(\exp(2\pi\sqrt{-1}a_1)t, -a_1, \hat{\phi}_2(t)) \mid \exp(2\pi\sqrt{-1}a_1)t \in U_0, -a_1 \in U_1\}.\end{aligned}$$

On the other hand, for $(a_0, 0, a_2, v) \in S'$

$$\mathrm{Gr}^W_{-1}((a_0, 0, a_2, v)) = (a_0, 0, a_2).$$

Then $(a_0, 0, a_2) \in (p_1)^{-1}(\phi(t))$ if, and only if, $a_0 = t$ and $a_2 = \hat{\phi}_2(t)$. □

Lemma 5.3.

$$(\Delta \times S') \cap K_{\sigma'} = \left\{ \left(t, (t, 0, \hat{\phi}_2(t), v)\right) \;\middle|\; \begin{array}{l} t \in U_0 \cap \Delta, \\ v \in \mathrm{Ker}(N) \cap U_3 \textit{ if } t = 0, \\ v \in U_3 \textit{ if } t \neq 0 \end{array} \right\}. \tag{5.3}$$

Proof. By Lemma 5.2, for $(a_0, 0, a_2, v) \in S'$

$$\phi(t) = p_1 \circ \mathrm{Gr}^{-1}_W((a_0, 0, a_2, v)) \Rightarrow a_0 = t \text{ and } a_2 = \hat{\phi}_2(t).$$

By Proposition 3.5, if $t \neq 0$, then $(t, 0, \hat{\phi}_2(t), v) \in S'$ for $v \in U_3$. If $t = 0$, since $(0, 0, v) \in U_1 \times U_2 \times U_3$ corresponds to $\exp(X_v)F_{(\sigma',Z),\tilde{\phi}}$, then $(0, 0, 0, v) \in S'$ for $v \in F^0_{\tilde{\phi}} + \mathrm{Ker}\,(N)$. Since $V \bigoplus F^0_{\tilde{\phi}} = H_{\mathbb{C}}$ by definition (2.4), $v \in \mathrm{Ker}\,(N)$. □

Homeomorphism. Let $S := W \cap (\Delta \times U_3)$ where W is in (2.5) and S is endowed with the strong topology in $\Delta \times U_3$. Then S is homeomorphic to (5.3). For the local trivialization (5.1), we get

$$\hat{\nu} : \Delta \to U_1 \times U_2 \times U_3 \subset \check{D}'; \quad t \mapsto (0, \hat{\phi}_2(t), 0).$$

Then we have an ANF

$$\nu : \Delta \to \Gamma' \backslash D'_{\Sigma'}; \quad t \mapsto p'_2 \circ p'_1(t, \hat{\nu}(t)).$$

Following (2.8) we set a neighborhood

$$\dot{S}(\nu) = \{((t, -\dot{v}), [\nu]_t) | (t, \dot{v}) \in \dot{S}\}$$

at $\alpha^{-1}(0, (\sigma', Z)) = ((0, 0), [\nu]_0)$ in J^{GGK} where $\dot{S}$ is the image of S in the quotient space $W/\sim$. Then $\alpha(\dot{S}(\nu))$ is the image of (5.3) through $p'_2 \circ p'_1$, which is a neighborhood of $(0, (\sigma', Z))$. In fact

$$\begin{aligned}\alpha((t, -\dot{v}), [\nu]_0) &= \begin{cases} (0, \exp(l(t)N')\exp(X_v)\hat{\nu}(t)) & \text{if } t \neq 0 \\ (\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_v)\hat{\nu}(0)) & \text{if } t = 0 \end{cases}\\ &= p'_2 \circ p'_1 \left(t, (t, 0, \hat{\phi}_2(t), v)\right).\end{aligned}$$

[KU, §3.1] gives a fundamental system of neighborhoods at $(0, 0)$ in S. This fundamental system of neighborhoods defines a fundamental system of neighborhoods at $((0, 0), [\nu]_0)$ in J^{GGK}, which goes to a fundamental system of neighborhoods at $(0, (\sigma', Z))$ in J^{KNU} through α. Therefore, α is a homeomorphism.

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